

STRONG LIMITS RELATED TO THE OSCILLATION MODULUS OF THE EMPIRICAL PROCESS BASED ON THE K-SPACING PROCESS

GANE SAMB LO

ABSTRACT. Recently, several strong limit theorems for the oscillation moduli of the empirical process have been given in the iid-case. We show that, with very slight differences, those strong results are also obtained for some representation of the reduced empirical process based on the (non-overlapping) k -spacings generated by a sequence of independent random variables (rv's) uniformly distributed on $(0, 1)$. This yields weak limits for the mentioned process. Our study includes the case where the step k is unbounded. The results are mainly derived from several properties concerning the increments of gamma functions with parameters k and one.

Nota-Bene. This paper was part of the PhD thesis at Cheikh Anta Diop University, 1991, not yet published in a peer-reviewed journal by August 2014. A slightly different version was published in *Rapports Techniques, LSTA, Université Paris VI*, 48, 1986, under the same title.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

Consider U_1, \dots, U_n a sequence of independent rv's uniformly distributed on $(0, 1)$, and let

$$U_{0,n} = 0 \leq U_{1,n} \leq \dots \leq U_{n,n} \leq U_{n+1,n} = 1$$

be their order statistics. The rv's

$$D_{i,n}^k = U_{ki,n} - U_{(i-1)k,n}, 1 \leq i \leq \left\lfloor \frac{n+1}{k} \right\rfloor = N,$$

where $[x]$ denotes the integer part of x , are called the non-overlapping k -spacings. Throughout, we shall assume that N and k are given and that n is defined by $n = \inf \left\{ j, \left\lfloor \frac{j+1}{k} \right\rfloor = N \right\}$ and then we will be able to study all our sequences as indexed by N since k will be either fixed or function of N .

Key words and phrases. Oscillation modulus, empirical processes, increments of functions, law of the iterated logarithm, order statistics.

The study of the properties of $D_{i,n}^k$ was introduced by Pyke [8] and several related papers have appeared in recent years (see e.g. [3]). One of the problem concerning the k-spacings is the study of the empirical process associated with $Nk D_{i,n}^k, 1 \leq i \leq N$.

In order to give a comprehensible definition of that process, we recall the following representation which can be found in [1] in the case where $(n+1)/k$ is an integer :

$$(1.1) \quad \{D_{i,n}^k, 1 \leq i \leq N\} =^d \left\{ \frac{Y_i}{S_{n+1}}, 1 \leq i \leq N \right\} =: \left\{ \frac{\left(\sum_{j=(i-1)k+1}^{j=ik} E_j \right)}{S_{n+1}}, 1 \leq i \leq N \right\},$$

where $=^d$ denotes the equality in distribution and S_n is the partial sum associated with E_1, \dots, E_n , a sequence of independent and exponential rv's with meanone, i.e., $S_n = E_1 + \dots + E_n$. Thus, it follows that, if $\frac{(n+1)}{k}$ is an integer, the limiting distribution function of $Nk D_{i,n}^k$, for any i and k fixed, is

$$H_k(x) = \int_0^x \frac{t^{k-1} e^{-t}}{(k-1)!} dt, \quad x \geq 0.$$

Therefore the empirical process (E.P.) associated with $Nk D_{i,n}^k, 1 \leq i \leq N$, may be defined by

$$(1.2) \quad \beta_N(x) = N^{\frac{1}{2}} \{F_N(x) - H_k(x)\}, \quad 0 \leq x \leq +\infty,$$

where F_N is the empirical distribution (E.D.F.) of $Nk D_{i,n}^k, 1 \leq i \leq N$, with

$$(1.3) \quad F_N(x) = \# \frac{\{i, 1 \leq i \leq N, Nk D_{i,n}^k \leq x\}}{N}, \quad x \geq 0.$$

Straightforward manipulations from (1.1), (1.2) and (1.3) as given in [1] show that even in the general case where $(N-1)k \leq n+1 \leq Nk$, the reduced process $\alpha_N(s) = \beta_N(H_k^{-1}(s)), 0 \leq s \leq 1$, satisfies

$$(1.4) \quad \{\alpha_N(s), 0 \leq s < 1\} =^d \left\{ N^{\frac{1}{2}} \{ \xi_N(\delta_n H_k^{-1}(s)) - s \} + 0 \left(N^{-\frac{1}{2}} \right), 0 \leq s \leq 1 \right\},$$

where H_k^{-1} is the inverse function of H_k , ξ_N is the E.D.F. pertaining to Y_1, \dots, Y_n and $\delta_n = \frac{S_{n+1}}{Nk}$.

The aim of this paper is to give the behavior of the oscillation modulus of $\alpha_N(\cdot)$ both where k is fixed and where $k \uparrow +\infty$. To this end we define

$$\wedge_N(a_N, R_N) = \sup_{0 \leq h \leq a_N} \sup_{0 \leq s \leq 1-h} |R_N(s+h) - R_N(s)|$$

and

$$k_N(a_N, R_N) = \frac{\wedge_N(a_N, R_N)}{(2a_N \log \log a_N^{-1})^{\frac{1}{2}}},$$

for any sequence of functions $R_N(s)$, $0 \leq s \leq 1$ and for any sequence $(a_N)_{N \geq 1}$, $0 < a_N < 1$. The properties of $\wedge_N(a_N, R_N)$, the oscillation modulus of R_N , have been first described by Csörgö and Révész [2] and Stute [10] when R_N represents the E.P. pertaining to a sequence of independent and uniformly distributed rv's with

$$(S1) \quad Na_N \rightarrow +\infty,$$

$$(S2) \quad \frac{(\log a_N^{-1})}{(Na_N)} \rightarrow 0$$

and

$$(S3) \quad \frac{(\log a_N^{-1})}{\log \log N} \rightarrow +\infty$$

as $N \rightarrow +\infty$.

Later, Mason, Shorack and Wellner (MSW) [7] dealt with the same for several choices of (a_N) and give among the results an Erdős-Rényi law.

The chief achievement of this paper is the extension of those limit results to some sequence of process $\bar{\alpha}_N$ equal in distribution to α_N . In fact, the fundamental role is played here by the properties of the tails of the gamma function $H'_k(\cdot)$, the derivative function of H_k . These properties are established in Section 2 through technical lemmas and the proofs of the following results are given in Section 3.

Theorem 1. *Let k be fixed. Then, there exists a sequence of processes $\alpha_N^-(s), 0 \leq s \leq 1, N = 1, 2, \dots$ such that*

$$(1.5) \quad \forall N \geq 1, \{\alpha_N(s), 0 \leq s \leq 1\} d_+ = \{\alpha_N^-(s), 0 \leq s \leq 1\}.$$

(I) *If $(a_N)_{N \geq 1}$ is a sequence of non-decreasing numbers satisfying the Csörgö-Révész-Stute conditions (S1), (S2) and (S3), then*

$$\lim_{N \uparrow +\infty} k_N(a_N, \alpha_N^-) = 1, a.s.$$

(II) *If*

$$a_N = cN^{-1} \log N, c > 0, N \geq 1$$

then

$$\lim_{N \uparrow +\infty} k_N^\circ(a_N, \alpha_N^-) = \left(\frac{c}{2}\right)^{\frac{1}{2}} (\beta^+ - 1), a.s., \text{ where } \beta^+ > 1$$

and

$$\beta^+ (\log \beta^+ - 1) = c^{-1} - 1.$$

(III)

$$I f a_N = (\log N)^{-c}, c > 0,$$

then

$$c^{\frac{1}{2}} \leq \lim_{N \rightarrow +\infty} \inf k_N(a_N, \alpha_N^-) \leq \lim_{N \rightarrow +\infty} \sup k_N(a_N, \bar{\alpha}_N) \leq (1+c)^{\frac{1}{2}}, a.s.,$$

(IV) *If*

$$a_N = c_N N^{-1} \log N, c_N \rightarrow 0$$

such that

$$(c_N \log N) = N a_N \rightarrow +\infty$$

and

$$(\log N)^{-1} (\log c_N^{-1}) \log \log N \rightarrow 0 \text{ as } N \uparrow +\infty,$$

then

$$\lim_{N \uparrow +\infty} \sup \frac{N^{\frac{1}{2}} \log \left(\frac{1}{c_N}\right)}{\log N} \wedge_N(a_N, \bar{\alpha}_N) \leq 2, a.s.$$

We also have

Theorem 2. *If $k = k(N) \rightarrow +\infty$ such that for some $\delta > 2$ and for some N_0 ,*

$$0 < a_N \leq t_k(\delta) = k^{k(\delta-2)} \exp\left(\frac{-k^\delta}{2}\right), N \geq N_0,$$

then Parts (I), (II), (III) and (IV) remain true.

Remark 1. *If each α_N is the spacings empirical process based on a sample depending on N , say $\chi(N)$, and if these samples $\chi(N)$, $N = 1, 2, \dots$ are mutually independent (this statistical situation is quite conceivable, for instance when checking homogeneity) the strong limit results of Theorem 1 are also valid for α_N . One might seek other conditions to get the same extensions. Here, we restrict ourselves to weak extensions in the following*

Corollary 1. *Let k be either fixed or $k \rightarrow +\infty$. Let $(a_N)_{N \geq 1}$ be a sequence of positive numbers such that $0 \leq a_N \leq t_k(\delta)$ when $N \geq N_0$, for some N_0 and $\delta > 2$. Then:*

(I) *Under the assumptions of Part I of Theorem 1, we have*

$$\lim_{N \rightarrow +\infty} k_N(a_N, \alpha_N) = 1 \text{ in probability.}$$

(II) *Under the assumptions of Part II of Theorem 1, we have*

$$\lim_{N \rightarrow +\infty} k_N(a_N, \alpha_N) = \left(\frac{c}{2}\right)^{\frac{1}{2}} (\beta^+ - 1) \text{ in probability.}$$

(III) *Under the assumptions of Part III of Theorem 1, we have*

$$\lim_{N \rightarrow +\infty} k_N(a_N, \alpha_N) = c^{\frac{1}{2}}, \text{ in probability.}$$

(IV) *Under the assumptions of Part IV of Theorem 1, we have*

$$\lim_{N \rightarrow +\infty} P\left(\frac{N^{\frac{1}{2}} \log\left(\frac{1}{c_N}\right)}{\log N}\right) \wedge_N(a_N, \alpha_N) > (2 + \varepsilon) = 0 \text{ for all } \varepsilon > 0.$$

Remark 2. *It appears from Theorems 1 and 2 that the oscillation modulus of $\bar{\alpha}_N$ and that of the uniform empirical process are almost the same. In [5], we prove that the exact strong bounds in (I) and (II) remain for α_N when a_N satisfies further conditions.*

Remark 3. *One might think that deriving the result of our corollary by using invariance principles (as given in [1] and [5]) and well-known results for the Brownian bridge would be easier (at least for some sequences a_N). This is not true at all (see Remark 4 below).*

2. TECHNICAL LEMMAS

It will follow from Lemma 1 of section 3 that the increments of $\bar{\alpha}_N$ behave as the increments of $\gamma_N(\psi(\cdot))$ and those of $\phi(\cdot)$ where $\gamma_N(\cdot)$ is the E.P. pertaining to

$$U_1, \dots, U_N, \psi(s) = H_k(\mu_n H_k^{-1}(s)), 0 \leq s \leq 1, \mu_n d = \delta_n,$$

$$n = 1, 2, \dots, \phi(s) = H'_k(H_k^{-1}(s)) H_k^{-1}(s), 0 \leq s \leq 1,$$

with $H'_k(x) = \frac{dH_k(x)}{dx}$, for all positive x . Then, since $k_N(\cdot, \gamma_N)$ is known, our study is reduced to describing the increments of $\psi(\cdot)$ and that of $\phi(\cdot)$, what we do in this paragraph.

Lemma 1. *Let k be fixed and $a = a_N$ be a sequence of positive numbers satisfying*

$$(Q1) \quad \left(n^{-1} \log \log n\right)^{\frac{1}{2}} \log \left(\frac{1}{a}\right) \rightarrow 0 \text{ as } N \rightarrow +\infty \text{ and } a \rightarrow 0,$$

then as $N \rightarrow +\infty$, we have the following properties

$$(2.1) \quad \sup_{0 \leq h \leq a} \sup_{0 \leq s \leq 1-h} |\psi(s+h) - \psi(s)| = a(1 + o(1)) \text{ a.s.},$$

uniformly in s ,

$$(2.2) \quad 0 \leq s \leq 1-a, |\psi(s+a) - \psi(s)| = a(1 + q(a)),$$

where $q(a) \rightarrow 0$, a.s., as $a \rightarrow 0$.

Proof of lemma 1.

We need several properties of gamma functions. First note that for a fixed k ,

$$(2.3) \quad s = 1 - H_k(s) = \frac{e^{-x} x^{k-1}}{(k-1)!} \left\{ 1 + \frac{k-1}{x} + \frac{k-2}{x^2} + \dots + \frac{(k-a)!}{x^{k-1}} \right\},$$

and

$$(2.4) \quad x = H_k^{-1}(1-s) = \log \left(\frac{1}{s} \right) - \log(k-1)! + (k-1) \log x \\ + \log \left(1 + \frac{k-1}{x} + \dots + \frac{(k-1)!}{x^{k-1}} \right)$$

from $k-1$ integrations by parts. Next for a fixed k or for $k \rightarrow +\infty$, we have, as $x \downarrow 0$,

$$(2.5) \quad s = H_k(x) = \frac{x^k}{k!} (1 + o(x)),$$

and

$$(2.6) \quad x = H_k^{-1}(s) = (k!)^{\frac{1}{k}} \left(1 + O\left(\frac{x}{k}\right) \right),$$

where for any function $g(\cdot)$, $g(x) = O(y)$ as $x \downarrow 0$ means that $\lim_{x \downarrow 0} \sup \left| \frac{g(x)}{y} \right| < +\infty$. To see this, use the following inequalities:

$$0 \leq t \leq x \Rightarrow e^{-x} \leq e^{-t} \leq 1,$$

to obtain that $e^{-x} \frac{x^k}{k!} \leq H_k(x) \leq \frac{x^k}{k!}$ and the results follow. Now, we are able to prove lemma A1.

Let us continue the proofs Lemma 1. Define

$$(2.7) \quad \Psi_h(s) = \psi(s+h) - \psi(s), 0 \leq s \leq 1-h, 0 \leq h \leq a, h = 1, 2, \dots$$

Straighforward computations give

$$(2.8) \quad \frac{d\Psi_h(s)}{ds} = \mu_n^{k-1} \left\{ \exp((\mu_n - 1) H_k^{-1}(s+h)) - \exp((\mu_n - 1) H_k^{-1}(s)) \right\}.$$

Thus, for each elementary event ω of the probability space, for each N (that is to say for each n) and for each h, $\Psi_h(\cdot)$ is non-decreasing of non-increasing according to the sign of $\mu_n(\omega) - 1$. Thus we have

$$(2.9) \quad \sup_{0 \leq s \leq 1-h} |\Psi_h(s)| = \max \{ |\Psi_h(0)|, |\Psi_h(1-h)| \}.$$

Computation of $\Psi_h(1-h)$. By using (2.3) and (2.4), with $h = 1 - H_k(x)$, we have

$$\begin{aligned} \mu_n H_k^{-1}(1-h) &= \mu_n \log \left(\frac{1}{h} \right) - \mu_n \log(k-1)! + \mu_n(k-1) \log x \\ &\quad + \mu_n \log \left(1 + \dots + \frac{(k-1)!}{x^{k-1}} \right) \end{aligned}$$

Now recall that $\Psi_h(1-h) = 1 - H_k(\mu_n H_k^{-1}(1-h))$ and, using (2.3), get

$$(2.10) \quad \Psi_h(1-h) = \mu_n^{k-1} \frac{\left(\log \left(\frac{1}{h} \right) \right)^{k-1} h^{\mu_n} ((k-1)!)^{\mu_n}}{(k-1)!} x^{-(k-1)\mu_n} (1 + q_1(h)),$$

where there exists A_k and B_k depending only on k (k being fixed) such that

$$(2.11) \quad |q_1(h)| \leq A_k x^{-1} \log x + B_k x^{-1}, \text{ as } h \rightarrow 0 \text{ (i.e. as } x \rightarrow +\infty).$$

These constants A_k and B_k are provided by the approximation

$$\left| \frac{\log\left(\frac{1}{h}\right)}{x} - 1 \right| \leq A_k x^{-1} \log x + B_k x^{-1},$$

as $h \rightarrow 0$, $x = H_k^{-1}(1-h) \rightarrow +\infty$. And (2.11) leads to

(2.12)

$$\Psi_h(1-h) = h^{MU_n} \left(\log\left(\frac{1}{h}\right) \right)^{(k-1)(1-\mu_n)} (1+q_2(h))^{-(k-1)\mu_n+1} (1+q_3(N)),$$

where $q_2(\cdot)$ satisfies (2.11) with the same constants A_k and B_k and $q_3(N) = o(1)$, a.s., independently of h , $0 \leq h \leq 1$, as $N \rightarrow +\infty$. Since the functions $x^{-1} \log x$ and x^{-1} are non-increasing as $x \rightarrow +\infty$, it follows from (2.10), (2.11) and (2.12) that

$$(2.13) \quad \forall 0 \leq h \leq a, \Psi_h(1-h) = (1+q(a)) h^{\mu_n} \left(\log\left(\frac{1}{h}\right) \right)^{(k-1)\mu_n},$$

where $q(a) \rightarrow 0$ and $N \rightarrow +\infty$. By convention, we shall write $g(h) = q(a)$ for $0 \leq h \leq a$, for all h , $0 \leq h \leq a$, $g(h) = o(1)$ where the " $o(1)$ " depends only on a , as $a \rightarrow 0$.

Computation of $\Psi_n(0)$.

We have $\Psi_h(0) = H_k(\mu_n H_k^{-1})$. Then by using (2.5)-(2.6), we obtain

$$(2.14) \quad \mu_n H_k^{-1}(h) = \mu_n (k!)^{\frac{1}{k}} h^{\frac{1}{k}} (1+q(a)), 0 \leq h \leq a.$$

Use again (2.5)-(2.6) and get $H_k(\mu_n H_k^{-1}(h)) = h \mu_n^k (1+q(a))$, a.s., $0 \leq h \leq a \rightarrow 0$, since k is fixed and $\mu_n \rightarrow 1$, a.s., as $N \rightarrow +\infty$. Then,

(2.15)

$$\sup_{0 \leq h \leq a} \sup_{0 \leq s \leq 1-h} \Psi_h(s) = (1+q(a)) \sup_{0 \leq h \leq a} \max \left\{ h, h^{\mu_n} (\log h^{-1})^{(k-1)(1-\mu_n)} \right\},$$

a.s., and $N \rightarrow +\infty$. But,

$$(2.16) \quad \forall N \geq 1, \frac{d \left\{ h^{\mu_n} (\log h^{-1})^{(k-1)(1-\mu_n)} \right\}}{dh} = h^{\mu_n} (\log h^{-1})^{(k-1)(1-\mu_n)}$$

$$(2.17) \quad \times \left\{ \mu_n - \frac{(k-1)(1-\mu_n)}{\log h^{-1}} \right\}.$$

Thus $h^{\mu_n} (\log h^{-1})^{(k-1)(\mu_n-1)}$ is non-decreasing when n sufficiently large since $k(1-\mu_n) \rightarrow 0$, a.s., as $N \rightarrow +\infty$ by the strong law of large numbers (k being fixed). Then,

$$(2.18) \quad 0 \leq h \leq a \Rightarrow h^{\mu_n} (\log h^{-1})^{(k-1)(1-\mu_n)} \leq a^{\mu_n} (\log a^{-1})^{(k-1)(1-\mu_n)},$$

a.s., as $N \rightarrow +\infty$. Furthermore,

$$(2.19) \quad (\log a^{-1})^{(k-1)(1-\mu_n)} = \exp((k-1)(1-\mu_n) \log \log a^{-1}) = 1 + o(1),$$

a.s., whenever $(1-\mu_n) \log \log a^{-1} \rightarrow 0$, a.s. But this is implied by (Q1). Indeed, we have by the law of the iterated logarithm (the loglog law) that

$$(2.20) \quad \lim_{N \rightarrow +\infty} \sup (2n^{-1} \log \log n)^{\frac{1}{2}} |\mu_n - 1| \leq 1, \text{ a.s.}$$

This together with (Q1) imply that $(1-\mu_n) \log \log a^{-1} \rightarrow 0$, a.s., as $N \rightarrow +\infty$. In fact, the loglog law holds for δ_n , that is

$$(2.21) \quad \lim_{N \rightarrow +\infty} \sup (2n^{-1} \log \log n)^{\frac{1}{2}} |\delta_n - 1| \leq 1, \text{ a.s.}$$

But (2.21) may be obtained from (see [5], Appendix)

$$\sum_{p \geq 0} P \left(\bigcup_{n_p}^{n_{p+1}-1} \left\{ (2n^{-1} \log \log n) |\delta_n - 1| \geq 1 + \frac{\varepsilon}{2} \right\} \right) < +\infty,$$

where (n_p) is an increasing and unbounded sequence of positive integers and $\varepsilon > 0$ is arbitrary. This and the equality in distribution of δ_n and μ_n for each N imply (2.20). The same *loglog-law* shows that (Q1) implies that

$$(2.22) \quad a^{1-\mu_n} = \exp((1-\mu_n) \log a^{-1}) = 1 + o(1) \text{ as } N \rightarrow +\infty.$$

We finally get from (2.13), (2.18), (2.19) and (2.22) that

$$(2.23) \quad \sup_{0 \leq h \leq a} \sup_{0 \leq s \leq 1-h} |\Psi_h(s)| = a(1 + o(1)) \text{ a.s., as } N \rightarrow +\infty,$$

which proves Part (2.1) of Lemma 1. To prove part (2.2), it suffices to remark that we may have through (2.8) that

$$\min(\phi_a(0), \phi_a(1-a)) \leq \phi_a(s) \leq \max(\phi_a(0), \phi_a(1-a)), 0 \leq s \leq -a,$$

and the part in question follows since the first part implies that $\phi_a(0) = a(1 + o(1))$, a.s. and $\phi_a(1-a) = a(1 + o(1))$, a.s., as $N \rightarrow +\infty$.

Lemma 2. *Let k be fixed, then we have as $a \rightarrow 0$, $N \rightarrow +\infty$,*

$$\sup_{0 \leq h \leq a} \sup_{0 \leq s \leq 1-h} |\phi(s) - \phi(s+h)| = (a \log a^{-1}) (1 + o(1)).$$

Proof of Lemma 2.

Consider $\Phi_h(s) = \phi(s+h) - \phi(s)$, $0 \leq s \leq 1-h$. Direct considerations yield that

$$\frac{d\Phi_h(s)}{ds} = H_k^{-1}(s) - H_k^{-1}(s+h), 0 \leq s \leq 1-h.$$

Then for each h , $\Phi_h(\cdot)$ is non-increasing and thus,

$$\sup_{0 \leq s \leq 1-h} |\Phi_h(s)| = \max\{|\Phi_h(0)|, |\Phi_h(1-h)|\}.$$

But, by (2.5)-(2.6),

$$\Phi_h(0) = H'_k(H_k^{-1}(h)) H_k^{-1}(h) = kh(1 + q(a)),$$

$0 \leq h \leq a \rightarrow 0$. Here we omit the details concerning the uniform approximations which provide $q(\cdot)$. These details are very similar to those of the computation of $\Phi_h(0)$. By the considerations that were previously used for getting (2.10) from (1.1), we have

$$\Phi_h(1-h) = H'_k(H_k^{-1}(1-h)) H_k^{-1}(1-h) = h \log h^{-1} (1 + q(a)), 0 \leq h \leq a \rightarrow 0.$$

Notice that $H'_k(H_k^{-1}(1-h))$ yields something like (2.10) while $H_k^{-1}(1-h)$ yields $(\log h^{-1})(1 + q(a))$, $0 \leq h \leq a \rightarrow 0$. We obtain

$$\sup_{0 \leq h \leq a} \sup_{0 \leq s \leq 1-h} |\Phi_h(s)| = 1 + q(a) \sup_{0 \leq h \leq a} \max(kh, h \log h^{-1}) = (1 + q(a)) (a \log a^{-1}),$$

$a \rightarrow 0$, since k is fixed here. Hence Lemma 2 is proved.

Now, we concentrate on the case where $k \rightarrow +\infty$. First, we give the following

Proposition 1. *Let*

$$0 \leq s \leq t_k(\delta) = k^{k(\delta-2)} \exp\left(-\frac{1}{2}k^\delta\right),$$

Then, as $k \rightarrow +\infty$, we have

$$(2.24) \quad x = H_k^{-1}(1-s) = (\log s^{-1})(1+q_4(s)),$$

where there exist A and k_0 such that $|q_4(s)| \leq A \frac{\log k}{k^{\delta-1}}$ for all $0 \leq s \leq t_k(\delta)$, $k \geq k_0$.

Proof. Integrating by parts, we get

$$\forall x \geq 0, \frac{x^{k-1}e^{-x}}{(k-1)!} \leq 1 - H_k(x) \leq \frac{x^{k-1}e^{-x}}{(k-1)!} + \frac{k}{x}(1 - H_k(x))$$

Then,

$$(2.25) \quad \frac{x^{k-1}e^{-x}}{(k-1)!} \leq 1 - H_k(x) \leq \left\{1 - \frac{k}{x}\right\}^{-1} \frac{x^{k-1}e^{-x}}{(k-1)!}, \forall x \geq 0.$$

We are able to see that the expansion of $H_k(x)$ is then possible if $k/x \rightarrow 0$. Now, let $0 \leq s \leq s_k = 1 - H_k(k^\delta)$. Apply (2.25) and get

$$0 \leq s \leq s_k \Rightarrow s = 1 - H_k(x) = \frac{x^{k-1}e^{-x}}{(k-1)!} \{1 + q_5(x)\},$$

with $|q_5(x)| \leq (1 - k^{-(\delta-1)})k^{-(\delta-1)}$ for all $0 \leq s \leq s_k$. But

$$s_k = \frac{k^{\delta(k-1)}e^{-k}}{(k-1)!} (1 + o(k^{-(\delta-1)})).$$

Then by Sterling's formula and some straightforward calculations, it is possible to find a k_1 such that $t_k(\delta) = k^{k(\delta-2)} \exp(-\frac{1}{2}k^\delta) \leq s_k$ for all $k \leq k_1$. Then for $0 \leq s \leq s_k, k \geq k_1$,

$$(2.26) \quad x = H_k^{-1}(1-s) = (\log s^{-1}) - \log(k-1)! + (k-1) \log x + o(q_5(s)).$$

Now since

$$0 \leq s \leq s_k \Rightarrow \left| \frac{(k-1) \log x}{x} \right| \leq \frac{\log k}{k^{\delta-1}} = o(k^{-(\delta-1)} \log k),$$

$$0 \leq s \leq s_k \Rightarrow \left| \frac{\log(k-1)!}{x} \right| \leq \frac{\log k!}{k^\delta} = o(k^{-(\delta-1)} \log k),$$

by Sterling's formula. Thus, these two facts and (2.26) together imply that

$$\forall s \leq s_k, \log s^{-1} = x(1 + o(k^{-(\delta-1)} \log k)) = H_k^{-1}(1-s)(1 + o(k^{-\delta+1} \log k)),$$

which was to be proved. We finally give two lemmas which correspond to Lemmas 1 and 2 in the case of infinite steps k . \square

Lemma 3. *Let k satisfy, as $N \rightarrow +\infty$,*

$$(K) \quad kN^{-1} (\log \log n) \rightarrow 0$$

and

$$(Q2) \quad k = k(N) \rightarrow +\infty$$

Then the following assertions hold.

$$(2.27) \quad \sup_{0 \leq h \leq a} \sup_{0 \leq s \leq 1-h} |\psi(s+h) - \psi(s)| = a(1 + o(1)), \text{ a.s., as } N \rightarrow +\infty.$$

$$(2.28) \quad |\psi(s+h) - \psi(s)| = a(1 + q(a)), \text{ for } 0 \leq s \leq 1-a, \text{ with } q(a) \rightarrow 0, \text{ a.s., as } N \rightarrow +\infty.$$

Proof of of Lemma 3. As in Lemma 1, we have

$$(2.29) \quad \sup_{0 \leq s \leq 1-h} |\psi_h(s)| = \max \{ |\psi_h(0)|, |\psi_h(1-h)| \}.$$

First we treat $\psi_h(0) = H_k(\mu_n H_k^{-1}(h))$. Equations (2.5)-(2.6) yield

$$\mu_n H_k^{-1}(h) = (k!)^{\frac{1}{k}} \left(1 + o \left(H_k^{-1} \frac{(a)}{k} \right) \right) \mu_n, 0 \leq h \leq a.$$

Now we note that $0 \leq s \leq a$ implies that $0 \leq H_k^{-1}(h) \leq C_1 a^{\frac{1}{k}} (k!)^{\frac{1}{k}}$ for small values of a , C_1 being a constant. Sterling's formula then implies for large values of k ,

$$0 \leq s \leq t_k, 0 \leq H_k^{-1}(h) \leq \text{Const. } k^{\delta-1} \exp \left(-\frac{1}{2} k^{\delta-1} \right).$$

Then $H_k^{-1}(h) \rightarrow 0$ and we are able to use (2.5)-2.6 to get

$$\forall 0 \leq h \leq a \Rightarrow H_k(\mu_n H_k^{-1}(h)) = \mu_n^k h (1 + o(H_k^{-1}(a))).$$

The *loglog-law* implies that

$$k(1 - \mu_n) = 0 \left(k(2n^{-1} \log \log n)^2 \right), \text{ a.s.}$$

Thus, whenever (K) is satisfied, one has

$$\mu_n^k = \exp(-k(1 - \mu_n)(1 + o(1))) \rightarrow 1, \text{ a.s.}$$

Hence

(2.30)

$$\forall 0 \leq s \leq a \leq t_k(\delta), \psi_h(0) = h(1 + q(a)) \leq a(1 + q(a)) = \psi_a(0).$$

We now treat $\psi_h(1 - h)$. By the proposition, we get

(2.31)

$$\forall 0 \leq h \leq a \leq t_k(\delta), X = \mu_n H_k^{-1}(1 - h) = \mu_n (\log h^{-1}) \left(1 + 0 \left(\frac{\log k}{k^{\delta-1}}\right)\right), a.s.$$

Since $\frac{X}{k} = 0(k^{-(\delta-1)}), a.s.$, one has

(2.32)

$$1 - H_k(X) = \mu_n^{k-1} \frac{(\log h^{-1})^{k-1}}{(k-1)!} \left(1 + 0 \left(\frac{\log k}{k^{\delta-2}}\right)\right) h^{\mu_n} x^{-(k-1)\mu_n} ((k-1)!)^{\mu_n} (1 + q_4(h)), a.s.,$$

as $N \rightarrow +\infty$. Replace x by $\log h^{-1}$ in (2.32). On account of (2.31) and of the fact that $\left(1 + 0(k^{1-\delta} \log k)^{k-1}\right) = \left(1 + 0(k^{2-\delta} \log k)\right)$, we get

$$\psi_h(1 - h) = ((k-1)!)^{1-\mu_n} (\log h^{-1})^{(k-1)(1-\mu_n)} h^{\mu_n} (1 + q(a)).$$

Finally, by taking (K) and (Q2) into account, we find ourselves in the same situation as in the proof of Lemma 1 (see Statement (2.12)). But in order to have the same conclusion, i.e.,

$$(2.33) \quad \sup_{0 \leq h \leq a} \psi_h(1 - h) = a(1 + q(a)), a.s., \text{ as } N \rightarrow +\infty,$$

we have to check that

$$((k-1)!)^{1-\mu_n} = \exp((1 - \mu_n) \log(k-1)!) =: \rho_n \rightarrow 1, a.s., \text{ as } N \rightarrow +\infty.$$

But the *loglog* - law and Sterling's formula together show that

$$\rho_n = \exp \left(0 \left(\left(\frac{k^{\frac{1}{4}} (\log \log n)^{\frac{1}{2}}}{N^{\frac{1}{2}}} \log a^{-1} \right) \left(\frac{\log k}{\log a^{-1}} \right) \right) \right).$$

Obviously the condition $0 < a \leq t_k(\delta)$ implies that $(\log k)/(\log a^{-1}) \rightarrow 0$ as $N \rightarrow +\infty$, and as $k \rightarrow +\infty$. This fact combined with (Q2) clearly shows that $\rho_n \rightarrow 1$ as $N \rightarrow +\infty$. Now, by putting together (2.29), (2.30) and (2.33), we get

(2.34)

$$(0 < a \leq t_k(\delta), \delta > 2) \Rightarrow \sup_{0 \leq a} \sup_{0 \leq s \leq 1-h} |\psi_h(s)| = a(1 + q(a)), a.s., \text{ as } N \rightarrow +\infty.$$

Lemma 4. *Let $0 < a \leq t_k(\delta), \delta > 2$. Then as $k \rightarrow +\infty$, we have*

$$\sup_{0 \leq h \leq a} \sup_{0 \leq s \leq 1-h} |\phi(s+h) - \phi(s)| = (a \log a^{-1}) (1 + q(a)), q(a) \rightarrow 0 \text{ as } N \rightarrow +\infty, a \rightarrow 0.$$

Proof of of Lemma 4. If we proceed as in Lemma A2 and as in Lemma A3, we get

$$\sup_{0 \leq h \leq a} \sup_{0 \leq s \leq 1-h} |\Phi_h(s)| = \max(k a, (a \log a^{-1})) (1 + q(a)), \text{ as } N \rightarrow +\infty, a \rightarrow 0$$

From there, the conclusion is obtained by noticing that the condition $0 < a \leq t_k(\delta)$ implies that $\frac{(\log a^{-1})}{k} \rightarrow +\infty$ as $k \rightarrow +\infty$.

3. PROOFS OF THE RESULTS

Throughout, we shall use the following representation which follows from [5] (see e.g. the study of $R_{N1}(x)$).

Lemma 5. *Let k be fixed of $k \rightarrow +\infty$ as $N \rightarrow +\infty$, then*

$$\begin{aligned} \{\alpha_N(s), 0 \leq s \leq 1\} &= \left\{ \gamma_N(\psi(s)) + N^{\frac{1}{2}} \{H_k(\mu_n H_k^{-1}(s)) - s\}, 0 \leq s \leq 1 \right\} \\ &=: \{\bar{\alpha}_N(s), 0 \leq s \leq 1, a.s.\} \end{aligned}$$

Lemma 5 will be systematically used. Then, if a_N satisfies

$$(3.1) \quad \lim_{N \rightarrow +\infty} \frac{(\log \log n)^2}{N a_N \log a_N^{-1}} = 0,$$

we will be able to focus our attention on $\gamma_N(\psi(s)) + N^{\frac{1}{2}}(\mu_n - 1)\phi(s)$ in the following way

$$\begin{aligned} (3.2) \quad \frac{\bar{\alpha}_N(s)}{b_N} &= \frac{\gamma_N(\psi(s))}{b_N} + \frac{N^{\frac{1}{2}}(\mu_n - 1)\phi(s)}{b_N} + b_N^{-1} O\left(N^{\frac{1}{2}} \log \log n\right), a.s. \\ &=: A_{N1}(s) + A_{N2}(s) + A_{N3}(s). \end{aligned}$$

with $b_N = (2a_N \log \log a_N^{-1})^{\frac{1}{2}} = b(a_N)$. It follows that if (Q3) holds we have $A_{N3}(s) = o(1)$, *a.s.*, uniformly with respect to s , $0 \leq s \leq 1$.

Proof of Part I of Theorem 1. By (2.34), we have

$$(3.3) \quad k_N(a_N, \bar{\alpha}_N) \leq k_N(a_N, A_{N1}) + k_N(a_N, A_{N2}) + k_N(a_N, A_{N3}),$$

and by Lemma 3, we have for a fixed k ,

$$k_N(a_N, A_{N2}) \leq N^{\frac{1}{2}} |1 - \mu_n| b(a_N) (1 + o(1)), a.s.,$$

as $N \uparrow +\infty$. Thus the *loglog-law* implies that

$$(3.4) \quad \lim_{N \rightarrow +\infty} k_N(a_N, A_{N2}) = o(1), a.s.,$$

whenever

$$(Q4) \quad k^{-\frac{1}{2}} (2 \log \log n)^{\frac{1}{2}} b(a_N) \rightarrow 0 \text{ as } N \rightarrow +\infty$$

is satisfied. On the other hand, Lemma 1 and Theorem 0.2 of Stute [10] together yield that

$$(3.5) \quad k_N(a_N, A_{N1}) \leq k_N(a_N, \gamma_N)(1 + o(1)) = 1 + o(1), a.s., \text{ as } N \rightarrow +\infty$$

Then if (Q1), (Q3), (Q4), (S1) and (S3) are satisfied, we get

$$\lim_{N \rightarrow +\infty} k_N(a_N, \bar{\alpha}_N) \leq 1, a.s.$$

Now let

$$\theta_N(a_N, R_N) = \sup_{0 \leq s \leq 1-a_N} \{R_N(s + a_N) - R_N(s)\}.$$

By Lemma 2, we have for large N that

$$(3.6) \quad b_N^{-1} \sup_{0 \leq s \leq 1-a_N} |A_{N2}(s + a_N) - A_{N2}(s)| \leq N^{\frac{1}{2}} |1 - \mu_n| b(a_N)(1 + o(1)), a.s.$$

Thus if (Q3) and (Q4) are satisfied, we get

$$(3.7) \quad \theta_N(a_N, \bar{\alpha}_N) \leq \theta_N(a_N, A_{N1}) + o(1), a.s., \text{ as } N \rightarrow +\infty.$$

Furthermore it may be derived from Theorem 0.2 of Stute [10] that (S1), (S2) and (S3) yield

$$(3.8) \quad b_N^{-1} \sup_{0 \leq s \leq 1-a_N} |\gamma_N(\psi(s) + o(a_N)) - \gamma_N(\psi(s) + a_N)| = o(1), a.s., \text{ as } N \rightarrow +\infty$$

It follows from (3.7) and (3.8) that (S1 - 2 - 3) and (Q1 - 3 - 4) together imply

$$b_N^{-1} \theta_N(a_N, \bar{\alpha}_N) \geq b_N^{-1} \left\{ \sup_{0 \leq s \leq 1-a_N} \gamma_N(\psi(s) + a_N) - \gamma_N(\psi(s)) \right\} + o(1) \text{ a.s.},$$

as $N \rightarrow +\infty$. Since $\psi : (0, 1 - a_N) \rightarrow (0, \psi(1 - a_N))$, is a bijection and since $\psi(1 - a_N) = 1 - a_N(1 + o(1)), a.s.$, we may use Lemma 1 (formulas (2.8) and (2.9) when (Q1) holds to find for any $\varepsilon > 0$, for any elementary event ω , an $N_o(\omega)$ such that

$$N > N_o \Rightarrow b_N^{-1} \theta_N(a_N, \bar{\alpha}_N) \geq \sup_{0 \leq s \leq 1-a_N(1+\varepsilon)} (\gamma_N(s + a_N) - \gamma_N(s)) + o(1), a.s.$$

Once again, we use the Theorem 02 of [10] to see that, under (S1-2-3), we have

$$\lim_{N \rightarrow +\infty} \sup_{0 \leq s \leq 1 - a_N(1+\varepsilon)} \left\{ \frac{|\gamma_N(s + a_N + \varepsilon a_N) - \gamma_N(s + a_N)|}{b_N} \right\} \leq \varepsilon^{\frac{1}{2}}, a.s.$$

Thus, under (Q1-3-4) and for large values of N , we get

$$(3.9) \quad b_N^{-1} \theta_N(a_N, \bar{\alpha}_N) \geq b_N^{-1} \theta_N(a_N, \gamma_N) - \left(1 + \frac{1}{2}\right) \varepsilon^{\frac{1}{2}}, a.s.,$$

Hence Lemma 2.13 in [10] and (3.9) together yield

$$(3.10) \quad \forall \varepsilon > 0, \lim_{N \rightarrow +\infty} \inf \theta_N(a_N, \bar{\alpha}_N) \geq (1 - \varepsilon)^{\frac{1}{2}} - \left(1 + \frac{1}{2}\right), a.s.$$

Finally (3.5) and (3.10) together ensure that

$$\lim_{N \rightarrow +\infty} k_N(a_N, \bar{\alpha}_N) = 1, a.s.,$$

whenever (Q1 – 3 – 4) and (S1 – 2 – 3) hold. But since $\log n \sim \log N$ (k being fixed), one has

$$(3.11) \quad \left(\frac{\log \log n}{n}\right)^{\frac{1}{2}} \log a_N^{-1} \sim k^{\frac{1}{2}} \left(\frac{\log \log N}{\log a_N^{-1}}\right)^{\frac{1}{2}} \frac{\left(a_N^{\frac{1}{2}} \log a_N^{-1}\right)^{\frac{1}{2}}}{\left(N^{\frac{1}{2}} a_N^{\frac{1}{2}}\right)},$$

$$(3.12) \quad \left(\frac{\log \log N}{N a_N \log a_N^{-1}}\right)^2 \log a_N^{-1} \leq \frac{(\log \log N)^2}{(\log a_N^{-1})^2},$$

$$(3.13) \quad (2 \log \log n) (a_N \log a_N^{-1}) \sim \left(\frac{\log \log N}{\log a_N^{-1}}\right) \left(a_N^{\frac{1}{2}} \log a_N^{-1}\right)^{\frac{3}{2}},$$

for large N . (2.1), (2.2) and (3.13) show that (S1) and (S2) imply (Q1-2-3) and this completes the proof of part I of Theorem 1.

Proof of Part II of Theorem 1.

The proof is the same as that of the first part. We only notice that if $a_N = cN^{-1} \log N$, $c > 0$, (Q1 – 3 – 4) are satisfied for a fixed k . To get Part II of Theorem 1, we use Theorem 1 (Part I) of [7] for the inequality " \leq " and the Erdős-Renyi law for the increments of the uniform empirical process due to Komlos *et al.*, and [4] for the inequality " \geq ". Similarly to the first case, we get an analogue to (3.9). That is, for any $\varepsilon > 0$, for any elementary event ω , we can find an $N_1(\omega)$ such that

$$\begin{aligned}
N > N_1 \Rightarrow k_N(a_N, \bar{\alpha}_N) &\geq \sup_{0 \leq s \leq 1-a(1+\varepsilon)} \left| \frac{\gamma_N(s+a(1+\varepsilon)) - \gamma_N(s)}{b_N} \right| \\
(3.14) \quad &- 2\varepsilon^{\frac{1}{2}} h(c\varepsilon) + \varepsilon^{\frac{1}{2}},
\end{aligned}$$

where for any s , $h(s) = \left(\frac{s}{2}\right)^{\frac{1}{2}} (\beta^+(s) - 1)$ and $\beta^+(s)$ is the unique solution of the equation $x(\log x - 1) + 1 = s^{-1}$ such that $\beta^+(s) \geq 1$. Now, since for any $f(\cdot), g(\cdot), K$,

$$\sup_{s \in K} \max(f(s), g(s)) = \max\left(\sup_{s \in K} f(s), \sup_{s \in K} g(s)\right) \text{ and } |x| = \max(x, -x)$$

and since (see e.g. the third formula that follows Statement 11 in [7])

$$\forall \varepsilon > 0, \lim_{N \rightarrow +\infty} \inf \frac{|\gamma_N(s+a(1+\varepsilon)) - \gamma_N(s)|}{b_N} = (1+\varepsilon)^{\frac{1}{2}} h((1+\varepsilon)c), a.s.,$$

then (3.14) implies that

$$\forall \varepsilon > 0, \lim_{N \rightarrow +\infty} \inf k_N(a_N, \bar{\alpha}_N) \geq (1+\varepsilon)^{\frac{1}{2}} h((1+\varepsilon)c) - 2^{\frac{1}{2}} h(c\varepsilon) - \varepsilon^{\frac{1}{2}}, a.s.$$

Thus it suffices to prove that : (2.1) for each fixed c , $h((1+\varepsilon)c) \rightarrow h(c)$ as $\varepsilon \rightarrow 0$, and : (2.2) for each fixed c , $\varepsilon^{\frac{1}{2}} h(c\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. But these two points may be directly obtained by simple considerations.

Proof of part III of Theorem 1.

The proof is very similar to that of Part I of Theorem 1. It suffices to remark that part III of Theorem 1 in [7] holds in the general case where $a_N = \alpha (\log N)^{-c}$, $c > 0, \alpha > 0$.

Proof of Part IV of Theorem 1.

Here $a_N = c_N N^{-1} \log N$, $c_N \rightarrow 0$ as $N \rightarrow +\infty$. Let $d_N = N^{\frac{1}{2}} (\log N)^{-1} \log c_N^{-1}$. On the one hand, we have

$$\begin{aligned}
(A) \quad & (N^{-1} \log \log N)^{\frac{1}{2}} \log a_N^{-1} \sim (N^{-1} \log \log N)^{\frac{1}{2}} \log c_N^{-1} + N^{-\frac{1}{2}} (\log \log N)^{\frac{1}{2}} \log N,
\end{aligned}$$

$$\frac{(\log \log n)^{\frac{1}{2}}}{c_N \log N} (\log a_N^{-1})^{-1} \sim \frac{(\log \log N)^{\frac{1}{2}}}{(\log c_N^{-1}) c_N \log N + c_N (\log N)^2 (1 + o(1))}$$

$$(B) \quad = \frac{(\log \log N)^{\frac{1}{2}}}{\log N} (c_N \log N)^{-1} \left(1 + \frac{\log c_N^{-1}}{\log N} + o(1) \right),$$

$$(C) \quad \left((\log \log n)^{\frac{1}{2}} (a_N \log a_N^{-1})^{\frac{1}{2}} \right)^2$$

$$(3.15) \quad = c_N (\log c_N^{-1}) N^{-1} (\log N) (\log \log N) + c_N N^{-1} (\log N)^2 (\log \log n) (1 + o(1)).$$

Obviously (A), (B) and (C) together imply that the conditions of Part IV of Theorem 1, namely, as $N \rightarrow +\infty$,

$$(W1) \quad c_N \rightarrow 0,$$

$$(W2) \quad c_N \log N \rightarrow +\infty$$

and

$$(W3) \quad (\log N)^{-1} (\log c_N^{-1}) (\log \log N) \rightarrow 0.$$

In turn these facts imply the conditions (Q1-3-4). On the other hand, we have

$$d_N \wedge_N (a_N, \bar{\alpha}_N) \leq d_N \wedge_N (a_N, A_{N1})$$

$$+ 0 \left(\frac{(a_N \log a_N^{-1}) (\log \log n)^{\frac{1}{2}} N^{\frac{1}{2}} \log c_N^{-1}}{k^{\frac{1}{2}} \log N} \right) + 0 \left(\frac{\log \log n}{\log N} \log c_N^{-1} \right),$$

a.s., as $N \rightarrow +\infty$, where we have used Lemma 2 and (3.2). Further, as $N \rightarrow +\infty$,

$$(Q5) \quad \frac{(a_N \log a_N^{-1}) (\log \log n)^{\frac{1}{2}} N^{\frac{1}{2}} \log c_N^{-1}}{k^{\frac{1}{2}} \log N} \sim \frac{(\log a_N^{-1}) (\log \log N) (\log N)^3}{(\log N) (k^{\frac{1}{2}} \log N) N} c_N \rightarrow 0,$$

by the definition of a_N and by (W1) and (W2). Thus, as $N \rightarrow +\infty$, we have

$$d_N \wedge_N (a_N, \bar{\alpha}_N) \leq d_N \wedge_N (a_N, A_{N1}) + o(1), a.s.$$

At this step, we apply Part II of Theorem 1 of [7] by using Lemma 1 which is true on account of (Q1).

Proof of Theorem 2.

We shall omit details of the proofs of the different parts that are the same as those of the parts of Theorem 1. The only problem concerns the bounds depending on k . However, this problem is solved by Lemmas 3 and 4. Hence we only provide the following remarks.

(R1) In our different choices of (a_N) , we have that $Na_N \rightarrow +\infty$, as $N \rightarrow +\infty$.

(R2) If $a_N \leq t_k(\delta)$, $\delta > 2$, then for any $y > 0$, there exists k_y such that

$$\forall k > k_y, \quad k^{-y}N \leq (Nt_k)^{-1} \leq (Na_N)^{-1} \rightarrow 0 \text{ as } N \rightarrow +\infty.$$

(R3) $(\log \log n) = (\log \log N)(1 + o(1))$ and $\log n = (\log N)(1 + o(1))$, as $N \rightarrow +\infty$.

With these remarks, it is easily seen, as in the proof of Theorem 1 that the conditions (K), (Q2), (Q3), (Q4) and (Q5) are satisfied at the same time with the specific assumptions of each part of Theorem 2 as follows.

(a) (Q2) and (K) are always satisfied if $a_N = t_k$. Indeed,

$$(3.16) \quad kN^{-1} \log \log n \sim \left(kN^{\frac{1}{2}}\right) \left(N^{-\frac{1}{2}} \log \log N\right) \rightarrow 0 \text{ as } N \rightarrow +\infty,$$

by (R2) and (K) and

$$(3.17)$$

$$k(\log \log n)(\log a_N^{-1}) \sim \left(N^{-\frac{1}{4}}k \log a_N^{-1}\right) \left(N^{-\frac{1}{4}} \log \log N\right) (Na_N)^{-\frac{1}{4}} \rightarrow,$$

by (R2) – (R3) and (Q2).

(b) In Parts I, II and III of Theorem 2, the implication $\{(S1), (S3)\} \Rightarrow \{(Q3), (Q4)\}$ is true whenever $\log \log n \sim \log \log N$ (see the lines that follow Formula (3.10)) and $(Na_N) \rightarrow +\infty$ as $N \rightarrow +\infty$, which are derived from (R1), (R2) and (R3).

(c) In Part IV, (Q5) is true independently of the behavior of k .

Thus we may use Lemmas 3 and 4 instead of Lemmas 1 and 2 in the proofs of Theorem 1 to get the results of Theorem 2 in the same way.

Proof of The Corollary.

This is a direct consequence of Theorems 1 and 2 and of Lemma 5. For Part III, the methods used in Part I of Theorem 1 must be repeated.

Remark 4. *By letting $(N \log \log N)^{\frac{1}{4}} (\log N)^{\frac{1}{2}} (2a_N \log a_N^{-1})^{-\frac{1}{2}} \rightarrow 0$, it would be possible to derive part I, III and IV of the Theorem 1 from invariance principles such as in [1] or [5]. But the necessary amount of work would be unchanged relatively to our method.*

REFERENCES

- [1] Aly, E.E.A., Beirlant, J. and Horv  th, L.(1984). Strong and weak approximation of the k-spacings processes. *Z. Wahrsch. verw. Gebiete*, 66, 461-484.
- [2] Cs  rg  , M. and R  v  sz, P. (1981). *Strong Approximation in Probability and Statistics*. Academic Press. New York.
- [3] Deheuvels, P. (1984). Spacings and applications. *Techn. report 13, LSTA*, Universit   Paris 6.
- [4] K  mlos, J. Major, M. and Tusn  dy, G. (1975). Weak convergence and embedding. In : *Colloquia Math. Soc. Janos. Boylai. Limit theorems of probability Theory*, 149-165. Amsterdam, North-Holland.
- [5] Lo, G.S. (1986). A strong upper bound in an improved approximation of the empirical k-spacings process and strong limits for the oscillation modulus. *Techn. report 47, L.S.T.A.*, Universit   Paris 6.
- [6] Lo  ve, M. (1963). *Probability Theory*. Van Nostrand Comp. Inc. Princeton., 3rd ed.
- [7] Mason, D.M., Shorack, G. and Wellner, A. (1983). Strong limit theorems for oscillation muduli of the empirical process. *Z. Wahrsch. verw. Gabiete*, 65, 83-87.
- [8] Pyke, R. (1972). Spacings. *J. Roy. Statist. Soc. Ser. B*, 27, 359-449.
- [9] Shorack, G.R. (1972). Convergence of quantile and spacing process with the application. *Ann. Math. Statist.*, 43, 1400-1411.
- [10] Stute, W. (1982). The oscillation behavior of empirical process. *Ann. Probab.*, 10, 86-107.